On the generation of surface waves by shear flows. Part 2

By JOHN W. MILES
Department of Engineering, University of California, Los Angeles

(Received 19 February 1959)

A previous analysis for the generation of surface waves by a parallel shear flow (Miles 1957*a*) is extended by: (*a*) presenting results based on a more accurate solution of the differential equation; (*b*) imposing the boundary condition at the surface wave, rather than at the mean surface; and (*c*) including the dominant viscous term in the complete Orr–Sommerfeld equation. The modification (*a*) yields an energy transfer somewhat smaller than that predicted previously but of the same order of magnitude as, and in rather better agreement with, observation, while (*b*) has no effect and (*c*) only a small effect for gravity waves. The analysis is based on the equations of motion in intrinsic co-ordinates (rather than the usual Orr–Sommerfeld equation) and may be of interest in other problems of hydrodynamic stability.

1. Introduction

This paper is a sequel to an earlier paper of the same title (Miles 1957*a*, hereinafter denoted by I). The principal result obtained there was that the mean rate at which energy is transferred from a parallel shear flow $U(y)$ to a surface wave of wavelength $2\pi/k$ and wave-speed c is given by

$$\overline{\dot{E}} = -\rho_a c(\pi U_c''/kU_c')\overline{v_c^2}, \tag{1.1}$$

where ρ_a denotes the density of the upper fluid (typically air), U_c'' and U_c' the curvature and slope of the wind profile at the point where $U = c$, and $\overline{v_c^2}$ the mean-square value of the vertical velocity there.†

The ultimate boundary-value problem posed in I was the solution of the inviscid Orr–Sommerfeld equation subject to boundary conditions at infinity and at the mean (undisturbed) position of the surface; this boundary-value problem was not actually solved, however, only an integral approximation for the energy-transfer coefficient having been obtained. The purpose of the following analysis is to: (*a*) present results based on an accurate integration of the differential equation (Miles & Conte 1959); (*b*) impose the boundary condition at the surface wave,

† In the notation of I, $\dot{E} = \zeta kc\overline{E}$, where $\overline{E}$ denotes the mean energy and $-\pi\zeta$ denotes the logarithmic decrement. The results then were presented in terms of a coefficient β, such that $\zeta = \sigma\beta(U_1/c)^2$, where $\sigma = \rho_a/\rho_w$ (air-water density ratio) and $U_1 = U_*/\kappa$ for a logarithmic profile. Prof. Lin has pointed out in a private communication that the energy transfer may be expressed as $\dot{E} = \tau c$, where $\tau = -\rho_a\overline{uv}$ is a Reynolds stress and u and v are the Cartesian components of velocity. Evaluating τ according to Lin (1954) then yields (1.1).

rather than at the mean surface; (*c*) include the dominant viscous term in the complete Orr–Sommerfeld equation. We shall find that: (*a*) yields an energy-transfer coefficient that is smaller than, but of the same order of magnitude as, that previously estimated; (*b*) has no effect on the end-results; and (*c*) shows that the viscous effects in the air just above the surface wave are small compared with those in the water (or other liquid), being of relative order $\sigma R_a^{-\frac{1}{2}} R_w$, where

$$R = c/k\nu, \tag{1.2a}$$

or, for a gravity wave,

$$R = c^3/g\nu \tag{1.2b}$$

denotes a Reynolds number based on wave-speed, wave-number, and the viscosity of either fluid (*a*: air or upper fluid; *w*: water or lower fluid).†

The model to be developed in § 2 resembles that of I in that it neglects perturbation Reynolds stresses (associated with the interaction between turbulent fluctuations in the original and perturbed flows; see I, Appendix); it differs in that it includes perturbation viscous stresses and is based on the intrinsic equations of motion (in which the streamlines appear as co-ordinate lines).‡ We shall include only a boundary-layer approximation to the perturbation viscous stresses, however, anticipating that (for large R) these stresses can be significant only in the small neighbourhoods of the surface wave (*outer viscous layer*) and of $U = c$ (*inner viscous layer*).

It might be objected that the neglect of the perturbation Reynolds stresses relative to perturbation viscous stresses is far more questionable than their neglect in the inviscid model of I, but to this objection one may reply that the primary purpose of including the viscous stresses is to show that they are indeed negligible compared with the terms included in I. We also remark that the outer viscous layer, which might have been suspected to be especially important in virtue of the interaction of viscous stresses in air and water, is likely to be confined within the laminar sublayer of the undisturbed flow.§ It is expedient, in this connexion, to introduce the dimensionless shear parameter

$$S_a = U'(0+)/kc = U_*^2/kc\nu_a = R_a(U_*/c)^2, \tag{1.3a, b, c}$$

where (1.3*b*) follows from (1.3*a*) through the equality of the shearing stresses $\rho_a U_*^2$ (U_* = Prandtl's friction velocity) and $\rho_a \nu_a U'(0+)$, while (1.3*c*) follows from (1.3*b*) through (1.2*a*). We find that the outer viscous layer will be confined within the laminar sublayer if (roughly) $S_a < 10$ and that this inequality will be satisfied for those combinations of parameters for which viscous dissipation in the

† The results of (*b*) and (*c*) were anticipated in I. See also Brooke Benjamin (1959).

‡ Following the completion of the present paper, the author spent a brief period at Cambridge University and learned of a very similar analysis by Brooke Benjamin (1959). His formulation, based on orthogonal curvilinear co-ordinates, is essentially equivalent to that given in § 2 below; his applications are more general, but there is very little overlap with the results obtained herein.

§ There is now a considerable body of evidence (see Takahashi 1958) that the flow *near* the water is aerodynamically smooth for wind-speeds as high as 800 cm/sec (at 400 cm above the water). This evidence does not include direct measurements in a laminar sublayer, but the measured profiles do imply the existence of a laminar sublayer having a thickness of the order of $3{\cdot}3\nu_a/U_*$.

air might have been expected to be significant (see § 5). We emphasize, on the other hand, that S_a may be rather large at higher wind-speeds.†

We shall develop the equations of motion for the water in § 3 on the assumption $S_w \ll 1$, where [based on $\rho_a U_*^2 = \rho_w \nu_w U'(0-)$]

$$S_w = U'(0-)/kc = \sigma U_*^2/kc\nu_w \tag{1.4a, b}$$

and

$$\sigma = \rho_a/\rho_w. \tag{1.5}$$

This permits the neglect of the shear flow in the water and the derivation of our results almost directly from Lamb (1945, § 349).

Having developed the equations of motion in §§ 2 and 3, we shall impose the boundary conditions of continuity of velocity and stress in § 4 to obtain a first approximation to the complex wave-speed. We assume, as in I, that the magnitude of the wave-speed is closely approximated by its unperturbed, inviscid value ($c^2 = g/k$ for gravity waves) and that this value may be used in the determination of the perturbation flows.

Numerical results based on our revised analysis are presented and discussed in § 5.

Concluding this introduction, we remark that the energy-transfer considered here augments that proposed by Phillips (1957), which considers the direct action of turbulent fluctuations in aerodynamic pressure on the water but neglects interaction between surface wave and air flow.‡ We hope to consider the simultaneous operation of these two complementary mechanisms in a subsequent paper.

2. Equations of motion for the air

We choose, as independent variables, the co-ordinates s and n measured along and normal to the streamlines (see figure 1) in a frame of reference moving with the wave-speed c and, as dependent variables, $q(s, n)$ and $\theta(s, n)$, the velocity along a streamline and the inclination of the streamline. (In accordance with the procedure outlined in § 1, c may be approximated as real throughout the following analysis except in (4.5e) and (4.6).) Starting from the intrinsic equations of motion (Milne-Thomson 1950, § 19.82)

$$qq_s + \rho^{-1}p_s = \nu q_{nn}, \tag{2.1a}$$

$$q^2\theta_s + \rho^{-1}p_n = 0, \tag{2.1b}$$

$$q_s + \theta_n q = 0, \tag{2.1c}$$

where ρ denotes density, p hydrodynamic pressure, ν kinematic viscosity (all parameters in this section referring to the upper fluid), subscripts partial differen-

† Brooke Benjamin (1959) has given an analysis for arbitrary values of S_a on the assumption that $U(y)$ is exactly linear, while Longuet-Higgins (1952) has treated a fixed wave ($c = 0$ or $S_a = \infty$) on the same assumption. Both find that the phase shifts associated with the viscous stresses can lead to a positive energy-transfer to the disturbance [cf. also Lin (1954)].

‡ It also seems likely that the energy-transfer associated with viscous phase shifts (see preceding footnote) could be of considerable significance for waves moving with speeds c corresponding to critical points ($c = U$) of small profile curvature. Such waves might not be important for the total energy-transfer (calculated in § 5 below), but could be important for the mean-square slope of the surface.

tiation and νq_{nn} the dominant shear term in a boundary-layer-type approximation, we seek the perturbation flow coupled with the displacement

$$\eta(s) = a\, e^{iks} \quad (k\,|a| \ll 1) \tag{2.2}$$

of the streamline $n = 0$ in a uniform, parallel shear flow $q^{(0)} = U(n) - c$. (Following the usual convention, the imaginary parts of complex quantities proportional to $\exp(iks)$ are to be discarded in the final interpretation.) The final motion will be unstable if $\mathscr{I}\{c\} > 0$, exhibiting the time-growth factor $\exp(k\mathscr{I}\{c\}\,t)$.

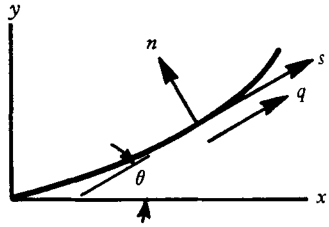

FIGURE 1. The co-ordinates for the intrinsic equations of motion (2.1a, b, c).

We first observe that the unperturbed solution to (2.1a,b,c) implied by our assumption of a strictly parallel shear flow is

$$q = q^{(0)}(n), \quad p = p^{(0)}(s), \quad \theta \equiv 0. \tag{2.3a,b,c}$$

In fact, we shall use (2.1a,b,c) to describe perturbations with respect to a turbulent flow for which $U(n)$ is the mean flow and in which the viscous stress $\rho\nu U'$ actually is balanced by a Reynolds stress; the model provided by (2.1a, b,c) then neglects perturbation Reynolds stresses (see I, Appendix).

We may linearize (2.1a,b,c) in the independent variable $\theta(s,n)$ by differentiating (2.1a) with respect to both s and n and (2.1b) twice with respect to s, taking the difference between the results to eliminate p, and eliminating q_s through (2.1c) whence

$$[(q^2\theta_n)_n + (q^2\theta_s)_s]_s = \nu(q\theta_n)_{nnn}. \tag{2.4}$$

Now, to first order in θ, we may approximate q by its undisturbed value $U(n) - c$ and assume θ to exhibit the harmonic s-dependence of (2.2), whence we obtain

$$[(U-c)^2\,\theta_n]_n - k^2(U-c)^2\,\theta = (\nu/ik)\,[(U-c)\,\theta_n]_{nnn} \tag{2.5}$$

as the linearized equation of motion. We remark that (2.5) differs from a boundary-layer approximation to the Orr–Sommerfeld equation (governing the perturbation stream function in Cartesian co-ordinates) in that it has a singularity at $U = c$; this implies that the linearized approximation to θ cannot be uniformly valid in the neighbourhood of $U = c$.† We shall find that this singularity introduces no essential difficulty (in so far as we require only the perturbation stresses at the interface $n = 0$), but it should be distinguished from the singularity that occurs at $U = c$ for the inviscid Orr–Sommerfeld equation of (2.9a) below; the latter singularity is a consequence of neglecting the viscous forces in a neighbourhood where the inertial forces tend to zero.

† Brooke Benjamin's (1959) formulation in orthogonal curvilinear co-ordinates avoids this difficulty but leads to an inhomogeneous form of the Orr–Sommerfeld equation unless terms in U^{iv} and U''' are neglected.

We shall delay consideration of the full complement of boundary conditions until § 4 below, but we note here that

$$\theta = \eta'(s) \quad (n = 0) \quad \text{and} \quad \theta \to 0 \quad (n \to \infty). \tag{2.6a, b}$$

We emphasize that these boundary conditions are imposed at the displaced, rather than the mean, position of the interface, thereby avoiding the assumption that the surface-wave displacement η must be small compared with a characteristic length (say c/U') for the shear profile; thus, we have only to assume $k|\eta| \ll 1$, rather than $U'(0+)|\eta|/c\,[= S_a k|\eta|] \ll 1$. It is for this reason that we choose a formulation in terms of $\theta(s, n)$, the streamline inclination in non-Cartesian co-ordinates, rather than the more conventional formulation in terms of a stream function in Cartesian co-ordinates.†

We shall seek asymptotic solutions to (2.5) as $R = c/k\nu \to \infty$. The formal procedure is essentially that for the Orr–Sommerfeld equation (Lin 1955, §§ 3.4 and 3.6) and yields two solutions that satisfy (2.6*b*). The first of these, the *inviscid solution*, may be obtained by setting $\nu = 0$ in (2.5); the second or *viscous solution* may be obtained by neglecting the second term on the left-hand side of (2.5) or, equivalently, omitting the pressure gradient in (2.1*a*) and disregarding (2.1*b*). We find it convenient to solve for $(U-c)\theta$ and $(U-c)\theta_n$ (which are proportional to vertical velocity and perturbation shearing stress) in these two cases and to separate the s-dependence by introducing the factor $\eta'(s)$; defining the dimensionless variables

$$\xi = kn, \quad f(\xi) = [U(n)-c]/c, \tag{2.7a, b}$$

we then write

$$f(\xi)\,\theta_1(s,n) = -ik\eta(s)\,\phi(\xi), \quad f(\xi)\,(\partial\theta_2/\partial n) = -ik^2\eta(s)\,\chi(R^{\frac{1}{3}}\xi), \tag{2.8a, b}$$

$$\theta = \theta_1 + \theta_2 = -ik\eta(s)\left[\frac{\phi(\xi)}{f(\xi)} + \int_\infty^\xi \frac{\chi(R^{\frac{1}{3}}\xi)\,d\xi}{f(\xi)}\right], \tag{2.8c}$$

where

$$f\phi'' - (f'' + f)\,\phi = 0 \tag{2.9a}$$

and

$$(d^2\chi/d\xi^2) - iRf\chi = 0 \quad \text{or} \quad \chi'' - if\chi = 0 \tag{2.9b, c}$$

as may be confirmed either by substituting (2.8*c*) in (2.5) and allowing R to tend to infinity or through the approximations described in the preceding sentence.

The inviscid equation (2.9) is identical with the inviscid Orr–Sommerfeld equation considered in I, so that our introduction of intrinsic co-ordinates and imposition of the boundary condition (2.6*a*) at the displaced position of the interface have not altered the inviscid problem. The viscous equation (2.9*b*), on the other hand, differs from its counterpart in the asymptotic solution of the Orr–Sommerfeld equation in consequence of our choice of variables. The available

† We may compare our introduction of $\theta(s, n)$ with the von Mises transformation suggested (but not used) in I to allow the imposition of the boundary conditions at the displaced interface. We also note that if Cartesian co-ordinates x_1 and x_2 are introduced according to $x_1 = x$ and $x_2 = y + \int_{-\infty}^{x} \theta\, dx$, where y denotes the vertical distance of a given streamline above the interface in the undisturbed flow, then partial differentiations with respect to s and n are equivalent to partial differentiations with respect to x and y. Cf. also Brooke Benjamin (1959).

methods of asymptotic solution remain the same, nevertheless, and we may use the *WKB* approximation to obtain†

$$\chi \sim f^{-\frac{1}{4}} \exp\left\{-\int_{\xi_c}^{\xi} \sqrt{(iRf)}\, d\xi\right\} [1 + O(R^{-\frac{1}{3}})], \tag{2.10}$$

where $f(\xi_c) = 0$ and the phase of the radical is $\pm\frac{1}{4}\pi$ as $\xi \gtrless \xi_c$, the path of integration being indented under the branch point at $\xi = \xi_c$ (Lin 1955, § 3.4).

We remark that (2.10) is not uniformly valid near $\xi = \xi_c$, but that it suffices for our purpose in so far as $S \ll R^{\frac{1}{3}}$ (the condition that the inner and outer viscous layers be well separated),‡ a condition that will be satisfied for those combinations of parameters for which viscous dissipation in the air is most significant (albeit still small). Formally superior solutions may be constructed (cf. Lin, §§ 3.6 and 8.5), but they serve only to substantiate this conclusion.

It remains to express the perturbation stresses on the interface in terms of ϕ and χ. Neglecting terms of $O(R^{-1})$, in keeping with our boundary-layer approximation, we may calculate the normal stress from (2.1*a*, *c*), (2.3*a*, *b*), (2.8), and (2.9*b*) according to

$$p_{22} = -(p - p_0) \tag{2.11a}$$

$$= -(ik)^{-1}\rho[q^2\theta_n + \nu(q_{nn} - U'')] \tag{2.11b}$$

$$= -(ik)^{-1}\rho[q^2\theta_n - (\nu/ik)(q\theta_n)_{nn}] \tag{2.11c}$$

$$= \rho c^2(f\phi' - f'\phi)\, k\eta. \tag{2.11d}$$

Evaluating (2.11*d*) at $n = 0$, where $f = -1$ and $f' = S$ [see (1.3)], we write

$$(p_{22})_{n=0+} = -\varpi\rho c^2 k\eta\phi_0, \tag{2.12}$$

where

$$\varpi = (\phi_0'/\phi_0) + S = (\alpha + i\beta)(U_1/c)^2; \tag{2.13a, b}$$

the parameters α and β are defined as in I (where $\phi_0 \equiv 1$), and the subscript zero implies evaluation at $\xi = 0+$.

The tangential stress is given by (within the boundary-layer approximation)

$$p_{12} = \rho\nu(q_n - U') \tag{2.14a}$$

$$= -(ik)^{-1}\rho\nu(q\theta_n)_n \tag{2.14b}$$

$$= \rho c^2 R^{-\frac{1}{3}}\chi'(R^{\frac{1}{3}}\xi)\, k\eta. \tag{2.14c}$$

Substituting χ from (2.10) and setting $\xi = 0+$, we obtain

$$(p_{12})_{n=0+} = -e^{-\frac{1}{4}i\pi}\rho c^2\, k\eta\, R^{-\frac{1}{2}}\chi_0. \tag{2.15}$$

We remark that the viscous solution enters the calculation of the normal stress and the inviscid solution that of the tangential stress only through the boundary conditions, which relate ϕ_0 and χ_0.

3. Equations of motion for the water

We shall proceed on the assumption that the shear flow in the water (induced by the traction of the shear flow in the air) may be neglected. As stated in § 1, this will be a good approximation if $S_w \ll 1$, where S_w is defined by (1.4). (This has

† The error factor in (2.10) is referred to the exact solution of (2.5), not (2.9*b*).

‡ The parameter $R^{\frac{1}{3}} S^{-1}$ is similar to the parameter $Z^{\frac{3}{2}}$ (Lin 1955, § 3.5) in the plane Poiseuille stability problem.

been confirmed by an analysis similar to that of the preceding section.) We emphasize that this assumption does not preclude the existence of a surface current, since our velocities are defined relative to such a current; all that $S_w \ll 1$ does imply is that the water moves approximately uniformly with the surface current to a depth of the order of $1/k$ ($= \lambda/2\pi$). We add that the small shear flow that is present must be oppositely directed to the airflow—i.e. $U_w < 0$ if $U_a > 0$; it follows that $U-c$ could not vanish in $n < 0$ for a wave travelling downwind, whence the shear flow in the water could not transfer energy to the surface wave through the mechanism of I.†

Assuming small perturbations with respect to a uniform flow $-c$ (in our moving frame of reference), we may take over Lamb's (1945, § 349) solution for a surface wave of the form (2.2) moving over a viscous liquid. Converting Lamb's notation to that introduced in § 2 above (in particular $v = -c\theta$; also, we omit the hydrostatic pressure from p_{22}), we may pose the streamline inclination and perturbation stresses in the forms

$$\theta = (a\,e^{\xi} + b\,e^{\kappa\xi})\,ik\eta, \tag{3.1}$$

$$p_{22} = -\rho c^2[(1+2iR^{-1})\,a\,e^{\xi} + 2i\kappa R^{-1}\,b\,e^{\kappa\xi}]\,k\eta, \tag{3.2}$$

$$p_{12} = \rho c^2[2R^{-1}a\,e^{\xi} + (2R^{-1}-i)\,b\,e^{\kappa\xi}]\,k\eta, \tag{3.3}$$

$$\kappa = (1-iR)^{\frac{1}{2}}, \quad \mathscr{R}\{\kappa\} > 0, \tag{3.4a, b}$$

where a and b are constants to be determined ($A = iac$ and $C = -bc$ are the corresponding constants in Lamb's solution) by the boundary conditions at the interface, and ρ, ν, and R are to be evaluated for the water. We emphasize that the boundary-layer approximation is not applicable to the water (since the interface is approximately free for the heavy fluid below the interface) and that (3.1)–(3.4) are based on the full, linearized equations of viscous flow; subsequently, we shall assume R_w to be large, but this approximation has yet to be invoked.

4. Determination of wave-speed

We may infer the boundary conditions at the interface from the considerations that both θ_a and θ_w must be equal to the slope of the surface wave, that the velocity (or θ_n) be continuous, that the shear stress be continuous, and that the discontinuity in normal stress be prescribed—viz.

$$\theta_a = ik\eta, \quad \theta_w = ik\eta, \quad \Delta\theta_n = 0, \tag{4.1a, b, c}$$

$$\Delta p_{12} = 0, \quad \Delta p_{22} = L\eta, \tag{4.1d, e}$$

where Δ denotes a jump operator according to

$$\Delta(\quad) = (\quad)_{n=0+} - (\quad)_{n=0-} \tag{4.2}$$

and $L\eta$ the (static) restoring stress of the interface. We may relate the operator L to the inviscid wave-speed in the absence of the upper fluid according to (see I)

$$L\eta = \rho_w c_w^2 k\eta; \tag{4.3}$$

† Such an energy transfer would be predicted for a wave travelling upwind, but it generally would be much smaller than the energy absorbed by viscous dissipation, primarily because U''_w almost certainly would be very small at that depth where $U_w - c = 0$.

for gravity waves, we have simply

$$c_w^2 = g/k. \tag{4.4}$$

We need not pose the boundary conditions at infinity, assuming them to be satisfied implicitly.

Substituting (2.8*c*), (2.12), (2.15), (3.1)–(3.3), and (4.3) in (4.1*a*–*e*), setting $f(0) = -1$ and $f'(0) = S_a$, and cancelling common factors, we may place the results in the form

$$\phi_0 - e^{\frac{1}{4}i\pi} R_a^{-\frac{1}{2}} \chi_0 = 1, \tag{4.5a}$$

$$a + b = 1, \tag{4.5b}$$

$$\phi_0' + S_a \phi_0 + \chi_0 = a + \kappa b, \tag{4.5c}$$

$$-e^{-\frac{1}{4}i\pi} \sigma R_a^{-\frac{1}{2}} \chi_0 = 2R_w^{-1}(a+b) - ib, \tag{4.5d}$$

$$c_w^2 = c^2[(1 + 2iR_w^{-1})\, a + 2i\kappa R_w^{-1} b - \sigma\varpi\phi_0]. \tag{4.5e}$$

Solving (4.5*a*–*d*) for ϕ_0, χ_0, a, and b, and substituting the results in (4.5*e*), we obtain

$$c^2 = c_w^2\{1 - 4iR_w^{-1} + \sigma\varpi - \sigma(1-\varpi)^2 e^{\frac{1}{4}i\pi} R_a^{-\frac{1}{2}} + O[R_w^{-\frac{3}{2}},\ \sigma R_a^{-\frac{1}{2}} R_w^{-\frac{1}{2}},\ \sigma S_a R_a^{-1},\ \sigma^2]\}. \tag{4.6}$$

Substituting ϖ from (2.13*b*) and R_w and R_a from (1.2*b*) in (4.6) and neglecting higher-order terms, we obtain the damping ratio (see first footnote in § 1 above)

$$\zeta = \frac{2\mathscr{I}\{c\}}{\mathscr{R}\{c\}} = \sigma\beta\left(\frac{U_1}{c}\right)^2 - \frac{4g\nu_w}{c^3} - \sigma\left(\frac{g\nu_a}{2c^3}\right)^{\frac{1}{2}}\left[1 - 2(\alpha+\beta)\left(\frac{U_1}{c}\right)^2 + (\alpha^2 - \beta^2 + 2\alpha\beta)\left(\frac{U_1}{c}\right)^4\right], \tag{4.7}$$

where the three terms on the right-hand side represent the positive energy-transfer from the shear flow, the viscous dissipation in the water, and the viscous dissipation in the air. Approximating the bracketed terms by 1 (they will be only slightly in excess of 1 for typical c/U_1), we find that the viscous dissipation in the air will be less than 10 % of that in the water for $c < 52$ cm/sec ($\lambda = 17$ cm) and will exceed it only for $c > 240$ cm/sec ($\lambda = 360$ cm). We also observe that neither of the dissipation terms is numerically significant for $c > 100$ cm/sec ($\lambda > 64$ cm) and wind speeds sufficiently high ($U_* \sim 10$ cm/sec) to permit the achievement of near-maximum values of β.

5. Numerical results

It remains to determine the parameter ϖ, as defined by (2.13*a*, *b*). This requires the determination of the ratio ϕ_0'/ϕ_0 through the integration of (2.9*a*) subject to (2.6*b*)—or, better, $\phi' + \phi \to 0$, $\xi \to \infty$. Only an integral approximation to the parameter β was attempted in I, but (2.9*a*) has since been integrated numerically (Conte & Miles 1959) for the logarithmic profile (I, equation (5.3*b*))

$$U(n) = U_1 \log(n/z_0), \quad U_1 = U_*/\kappa, \tag{5.1a, b}$$

with $\kappa = 0{\cdot}4$. The results for α and β, defined as in I and (2.13*b*) above, are plotted *vs* ξ_c and c/U_1 in figures 2 to 6 for three values of the parameter Ω, where (I, equation (6.2))

$$\xi_c = \Omega(U_1/c)^2 e^{c/U_1}, \quad \Omega = gz_0/U_1^2, \tag{5.2a, b}$$

with $c^2 = g/k$ (in zero'th approximation). Figure 6 has been included to illustrate the phase shift between the perturbation pressures outside the critical layer and at the surface wave. We add that the independent variable actually used in the integration was f, rather than ξ, and the logarithmic profile was assumed to be

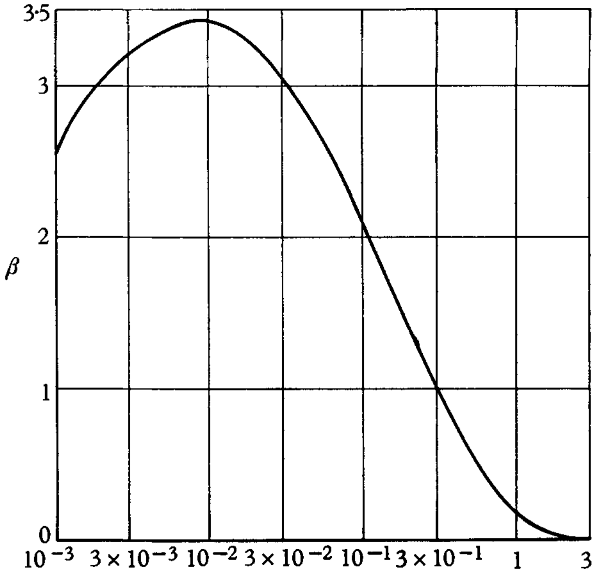

FIGURE 2. β *vs* ξ_c.

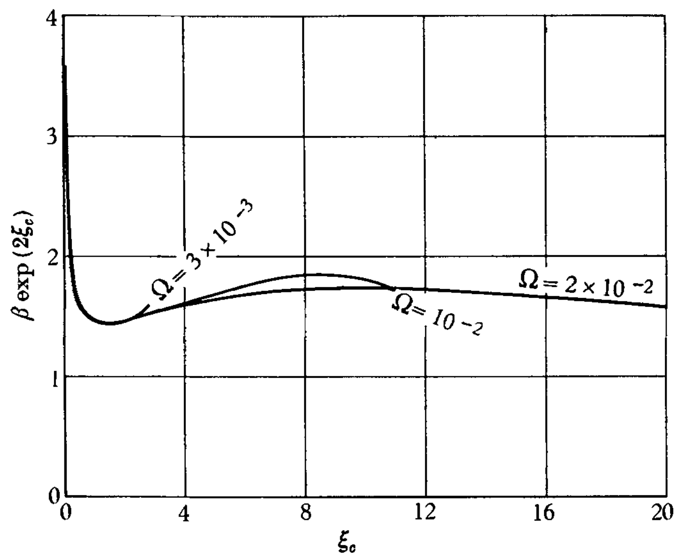

FIGURE 3. $\beta \exp(2\xi_c)$ *vs* ξ_c.

valid down to $f = -1$ (where $\xi = kz_0$). The correction required for the departure of the actual profile from the logarithmic profile in (say) $\xi < \xi_1$ is found in Appendix A to be $O(\xi_1)$ and therefore negligible for the range of the numerical integration.

Comparing the results of figures 2 and 3 with those of I (figure 1), we find that, as previously concluded, β *vs* ξ_c is essentially independent of Ω for $\xi_c < 2$ and is $O(e^{-2\xi_c})$ as $\xi_c \to \infty$; on the other hand, the numerical values of β are generally smaller than those estimated in I. We also find that α *vs* ξ_c is independent of Ω for

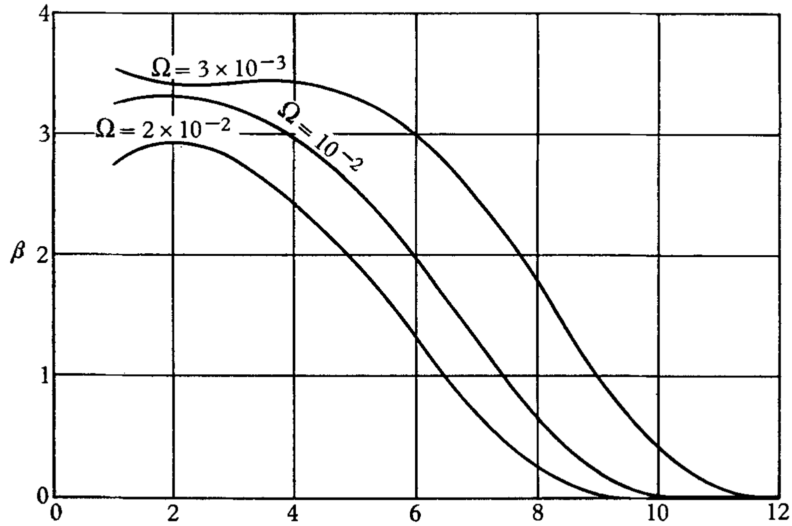

FIGURE 4. β *vs* c/U_1; $\Omega = gz_0/U_1^2$.

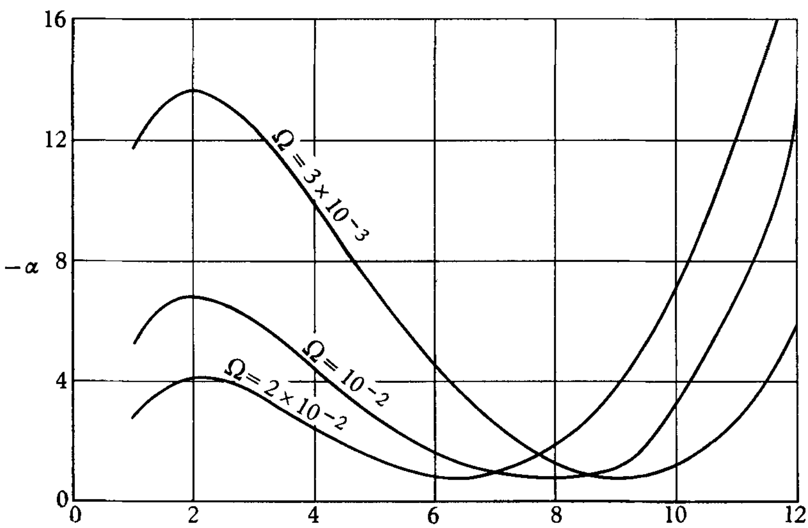

FIGURE 5. $-\alpha$ *vs* c/U_1.

sufficiently small ξ_c, since α then may be expressed in terms of β (see Appendix B). Assuming aerodynamically smooth flow ($z_0 = \nu_a/9U_*$), we find that the critical wind-speed at which the energy-transfer from the shear flow is just balanced by laminar dissipation in the water [see I, equations (6.5) to (6.7)] is given by

$U_1 = 14$–15 cm/sec for waves of length 20–30 cm. The availability of energy from the direct action of the turbulent fluctuations of the wind (as described by Phillips)† renders this critical wind-speed of only secondary significance, however, and there is now little reason to compare it directly with the minimum wind-speed ($U_1 = 9$–10 cm) at which water waves are first observed.

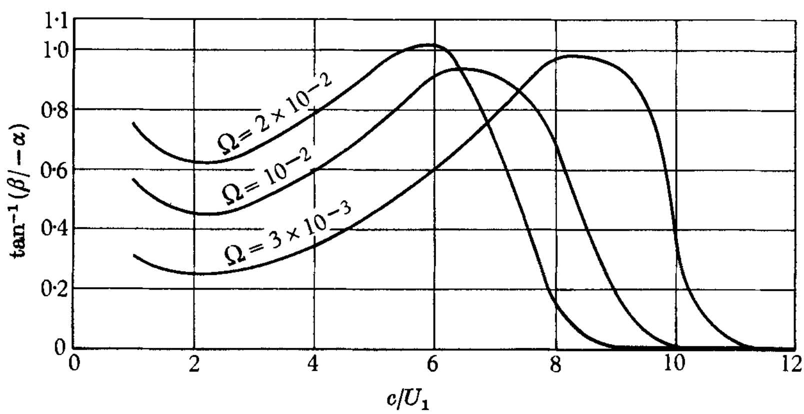

FIGURE 6. The phase shift $\tan^{-1}(\beta/-\alpha)$ *vs* c/U_1.

We conclude by calculating mean values of β, say β_E and β_τ, on the basis of mean energy-transfer and mean Reynolds stress. The rate at which energy is transferred from the shear flow to surface waves by normal pressures is given by

$$\overline{\dot{E}} = \overline{\mathscr{R}\{p_{22}(-ikc\eta)^*\}} \tag{5.3a}$$

$$= \rho_a U_1^2 \overline{(c\beta k^2 |\eta|^2 \cos^2\theta)} \tag{5.3b}$$

$$= \rho_a U_1^2 \int_0^\infty \int_{-\frac{1}{2}\theta_0}^{\frac{1}{2}\theta_0} c\beta\left(\frac{c}{U_1\cos\theta}\right) k^2 S(k,\theta)\cos^2\theta\, k\, dk\, d\theta, \tag{5.3c}$$

where θ now denotes the angle between the wind and the direction of propagation of a given wave, θ_0 the total beam width of the wave spectrum, and S the power spectral density of the surface displacement; we have substituted p_{22} from (2.12) with $\phi_0 = 1$ (corresponding to the neglect of the viscous solution), assumed that the effective wind-speed for an obliquely moving wave is reduced by $\cos\theta$, and posed β as a function of the ratio of wave-speed to effective wind-speed.‡ It follows from (5.3c) that we may define β_E, the mean value of β for energy transfer, according to

$$\int_0^\infty \int_{-\frac{1}{2}\theta_0}^{\frac{1}{2}\theta_0} c\left[\beta\left(\frac{c}{U_1\cos\theta}\right) - \beta_E\right] k^2 S(k,\theta)\cos^2\theta\, k\, dk\, d\theta = 0. \tag{5.4}$$

We shall assume the Neumann spectrum (I, equation (8.4))

$$k^2 Sk\, dk\, d\theta = C\, e^{-2(c/U_a)^2}\, dc\, d\theta, \tag{5.5}$$

† And perhaps also from the viscous phase shifts predicted by Longuet-Higgins and Brooke Benjamin (see last two footnotes in § 1 above).

‡ More precisely, $\beta = \beta(c/U_1\cos\theta, \Omega)$, where $\Omega = gz_0/U_1^2\cos^2\theta$. The subsequent calculations neglect the dependence of Ω on $\cos\theta$.

where C is a constant [irrelevant for (5.4)] and U_a the wind speed at the anemometer height. If we also introduce the change of variable $x = c/U_1 \cos\theta$, we may place the result for β_E in the form

$$\beta_E = 4\left(\frac{U_1}{U_a}\right)^2 \int_0^\infty x\beta(x) f_E\left[\left(\frac{U_1}{U_a}\right)^2 x^2, \theta_0\right] dx, \tag{5.6}$$

where
$$f_E(y,\theta_0) = \int_{-\frac{1}{2}\theta_0}^{\frac{1}{2}\theta_0} e^{-2y\cos^2\theta}\cos^4\theta\, d\theta \bigg/ \int_{-\frac{1}{2}\theta_0}^{\frac{1}{2}\theta_0} \cos^2\theta\, d\theta. \tag{5.7}$$

We shall consider the two extremes $\theta_0 = 0$ and π, for which (5.7) yields

$$f_E(y,0) = e^{-2y} \tag{5.8a}$$

and
$$f_E(y,\pi) = e^{-y}[I_0(y) - (1+\tfrac{1}{2}y^{-1})\, I_1(y)], \tag{5.8b}$$

where (5.8*b*) follows from (5.7) through the integral representation of I_n, the modified Bessel function of the first kind.

The mean Reynolds stress (cf. first footnote in § 1) in the direction of the wind is given by

$$\overline{\tau} = \overline{\mathscr{R}\{-p_{22}(ik\eta)^*\cos\theta\}}. \tag{5.9}$$

Following the development of (5.3) and (5.4), we then may define β_τ, the mean value of β for Reynolds stress, according to

$$\int_0^\infty \int_{-\frac{1}{2}\theta_0}^{\frac{1}{2}\theta_0} \left[\beta\left(\frac{c}{U_1\cos\theta}\right) - \beta_\tau\right] k^2 S(k,\theta)\cos^3\theta\, k\, dk\, d\theta = 0. \tag{5.10}$$

Introducing S from (5.5), we obtain†

$$\beta_\tau = \sqrt{\left(\frac{8}{\pi}\right)}\left(\frac{U_1}{U_a}\right)\int_0^\infty \beta(x) f_\tau\left[\left(\frac{U_1}{U_a}\right)^2 x^2, \theta_0\right] dx, \tag{5.11}$$

where
$$f_\tau(y,\theta_0) = \tfrac{1}{4}[\sin(\theta_0/2) - \tfrac{1}{3}\sin^3(\theta_0/2)]^{-1}(\theta_0 + \sin\theta_0) f_E(y,\theta_0). \tag{5.12}$$

Numerical integrations of (5.6) and (5.11) have been carried out for $\Omega = 10^{-2}$, which is representative of fully developed rough flow [see I, equation (7.5*a*, *b*)], and $U_a/U_1 = 9$, which is appropriate for wind-speeds of the order of 10 m/sec at 10 m above the surface.‡ It also was assumed that $\beta(x) \equiv 0$ for $x \leqslant 3$, where the logarithmic profile presumably ceases to hold.§ The resulting values of β_E are 1·24 and 1·05 for $\theta_0 = 0$ and π; the values of β_τ are 0·90 and 0·75. The corresponding

† Comparing (5.9) and (5.11) $\times (U_1/U_a)^2$ with I (8.1) and (8.7), we note that the latter results are not entirely consistent in their treatment of cos θ-factors [see also Munk (1955) on this point]; accordingly, $\beta_\tau = (U_a/U_1)^2 \beta_M$ only for $\theta_0 = 0$, but the discrepancies are relatively unimportant compared with other uncertainties in the numerical calculations.

‡ The wind-speed of 10 m/sec at 10 m is representative (in order of magnitude) of wind-speeds for which the Neumann spectrum was measured; for a wind-speed of 3 m/sec at 10 m $U_a/U_1 = 12$ would be closer to the mark.

§ A rough correction factor for the fall-off of profile curvature as a laminar sublayer is approached was constructed on the basis of the approximation $\beta \sim (U_c''/U_c')$ and an interpolated profile (linear at boundary and asymptotically logarithmic, Miles 1957*b*). This factor was found to be 0·99, 0·86, and 0·19 for $c/U_1 = 4{\cdot}8$, 3·6, and 2·9, respectively, indicating a rather sharp cut-off of β as the critical layer approaches the laminar sublayer. It is by no means certain that the flow *close to the surface* is aerodynamically smooth, but Takahashi's measurements (1958) tend to support such an assumption.

values referred to U_a—viz. $(U_1/U_a)^2\beta_E$ and $(U_1/U_a)^2\beta_T$—are $1\cdot5\times10^{-2}$, $1\cdot3\times10^{-2}$, $1\cdot1\times10^{-2}$ and $0\cdot9\times10^{-2}$. The latter values are in order-of-magnitude agreement with the 'sheltering coefficient' of $1\cdot3\times10^{-2}$ estimated by Sverdrup & Munk (1947) from their data on wave-growth and with the incremental shearing stress inferred by Munk (1955) from measurements made by Van Dorn. We emphasize that any of these numbers might be modified by a factor as large as (but probably no larger than) two either by changes in the assumed values of U_a/U_1, Ω and the cut-off point for $\beta(x)$ or by changes in the treatment of the observational data.

The results of the preceding paragraph are summarized in table 1.

$U(y) = U_1 \log(y/z_0) = U_1 \log(100\, gy/U_1^2)$, $U_a \doteqdot 9U_1$

θ_0	β_E	β_T
0	$1\cdot24\,(1\cdot5\times10^{-2})$	$0\cdot90\,(1\cdot1\times10^{-2})$
π	$1\cdot05\,(1\cdot3\times10^{-2})$	$0\cdot75\,(0\cdot9\times10^{-2})$

Sverdrup & Munk: $(U_1/U_a)^2\beta_E \doteqdot 1\cdot3\times10^{-2}$
Munk/Van Dorn: $(U_1/U_a)^2\beta_T \doteqdot 0\cdot6-1\cdot8\times10^{-2}$

TABLE 1. Comparison with observation.

6. Conclusions

We conclude that our model for energy-transfer from a parallel shear flow to deep-water gravity waves yields a total energy-transfer in order-of-magnitude agreement with observation. This suggests that such a mechanism may be a decisively important adjunct to initial excitations from other sources—in particular, turbulent fluctuations of wind pressure, as in the model proposed by Phillips (1957).

I am indebted to C. S. Cox, W. H. Munk and W. H. Van Dorn of the Scripps Institute of Oceanography for frequent discussions on the general subject of wave-generation, to M. S. Longuet-Higgins for making available to me his unpublished work, and to T. Brooke Benjamin for making available to me the page proof of his paper and for reading the manuscript of the present paper and offering several valuable suggestions for its improvement. I also take pleasure in acknowledging support, in the form of a fellowship, from the John Simon Guggenheim Memorial Foundation.

Appendix A

Suppose that the profile $f_l(\xi)$ agrees with the true profile $f(\xi)$ for $\xi > \xi_1$ but not for $\xi < \xi_1$ and that a value of ϖ, say ϖ_l, has been calculated by approximating f by f_l in $\xi < \xi_1$; then, given $f(\xi), f_l(\xi)$ and ϖ_l we require the true value of ϖ on the assumption that ξ_1 is small.

Let ϕ_1 and ϕ_1' denote the true values of $\phi(\xi)$ and $\phi'(\xi)$ at $\xi = \xi_1$; then we may integrate (2.9a) inward from ξ_1 to obtain the approximate solution (cf. Lin 1955, § 3.4)

$$\phi(\xi) = (\phi_1/f_1)\, f(\xi) + (\phi_1' f_1 - \phi_1 f_1')\, f(\xi) \int_{\xi_1}^{\xi} \frac{d\xi}{f^2(\xi)}, \tag{A 1}$$

with an error factor $1+O(\xi_1^2)$. Substituting (A 1) in (2.13a), we may place the result for ϖ in the form

$$\frac{1}{\varpi}=\frac{1}{\varpi_1}-\int_0^{\xi_1}\frac{d\xi}{f^2(\xi)}, \tag{A 2}$$

where

$$\varpi_1=\frac{f_1(\phi_1' f_1-\phi_1 f_1')}{\phi_1}. \tag{A 3}$$

Similarly,

$$\frac{1}{\varpi_l}=\frac{1}{\varpi_1}-\int_{\xi_0}^{\xi_1}\frac{d\xi}{f_l^2(\xi)}, \tag{A 4}$$

where ξ_0 denotes the lower limit actually used in the calculation based on f_l.

Assuming

$$f_l(\xi)=(U_1/c)\log(\xi/\xi_c),\quad f_l(\xi_0)=-1, \tag{A 5a, b}$$

introducing the logarithm as the variable of integration in (A 4), and eliminating ϖ_1 between (A 2) and (A 4), we obtain

$$\frac{1}{\varpi}=\frac{1}{\varpi_l}+\left(\frac{c}{U_1}\right)^2\xi_c\int_{-c/U_1}^{\log(\xi_1/\xi_c)}\frac{e^\eta\,d\eta}{\eta^2}-\int_0^{\xi_1}\frac{d\xi}{f^2(\xi)}. \tag{A 6}$$

The first integral may be reduced to the tabulated exponential integral; assuming $\xi_1<\xi_c$ (as must be so if the logarithmic profile is valid through the critical layer) and that f decreases monotonically from 0 at $\xi=\xi_c$ to -1 at $\xi=0$, we then find that both integrals are $O(\xi_1)$. Choosing the lower limit of validity for the logarithmic profile as $30z_0$, we have

$$\xi_1=30kz_0=30\Omega(U_1/c)^2, \tag{A 7}$$

which renders the difference between ϖ and ϖ_l negligible for the range of parameters covered in figures 2–6.

Appendix B

We seek to express α in terms of β for small ξ_c. Rewriting (2.9a) according to

$$(f\phi'-f'\phi)'=f\phi, \tag{B 1}$$

integrating both sides between $\xi=0$ and $\xi=\xi_c$, setting $f(0)=-1$ and $f'(0)=S$, and evaluating ϖ from (2.13), we obtain

$$(\alpha+i\beta)(U_1/c)^2=f_c'(\phi_c/\phi_0)+\phi_0^{-1}\int_0^{\xi_c}f\phi\,d\xi \tag{B 2a}$$

$$=f_c'(\phi_c/\phi_0)[1+O(\xi_c^2)]. \tag{B 2b}$$

Taking the absolute value of both sides of (B 2b) yields

$$\alpha^2+\beta^2=(c/U_1)^2f_c'^2\,|\phi_c/\phi_0|^2; \tag{B 3}$$

eliminating $|\phi_c/\phi_0|^2$ through [I (4.3)]

$$\beta=-\pi(c/U_1)^2(f_c''/f_c')\,|\phi_c/\phi_0|^2, \tag{B 4}$$

assuming $\alpha<0$ [as may be proved from I, equation (4.1)], and evaluating f_c' and f_c'' for the logarithmic profile of (A 5a), we obtain

$$\alpha=-[(\beta/\pi\xi_c)-\beta^2]^{\frac{1}{2}}. \tag{B 5}$$

We have used this result to obtain an independent check on the numerical integration for small ξ_c.

REFERENCES

BENJAMIN, T. BROOKE 1959 *J. Fluid Mech.* **6**, 161.

CONTE, S. & MILES, J. W. 1959 *J.S.I.A.M.* (in the Press).

LAMB, H. 1945 *Hydrodynamics*. New York: Dover.

LIN, C. C. 1954 *Proc. Nat. Acad. Sci., Wash.*, **40**, 741.

LIN, C. C. 1955 *The Theory of Hydrodynamic Stability*. Cambridge University Press.

LONGUET-HIGGINS, M. S. 1952 (Unpublished manuscript.)

MILES, J. W. 1957*a* *J. Fluid Mech.* **3**, 185.

MILES, J. W. 1957*b* *J. Aero. Sci.* **24**, 704.

MILNE-THOMSON, L. M. 1950 *Theoretical Hydrodynamics*. New York: Macmillan.

MUNK, W. 1955 *Quart. J. R. Met. Soc.* **81**, 320.

PHILLIPS, O. M. 1957 *J. Fluid Mech.* **2**, 417.

SVERDRUP, H. U. & MUNK, W. 1947 *Wind, Sea and Swell. Theory of Relations for Forecasting. Publ. Hydrog. Off., Wash.*, no. 601.

TAKAHASHI, T. 1958 *Micrometerological Observations and Studies over the Sea*. Faculty of Fisheries, Kagoshima University, Japan.

CORRIGENDA FOR I (MILES 1957*a*)

(1) Equations (3.2*a*, *b*) should read: $u = -\psi_y$, $v = \psi_x$.
(2) Footnote, p. 193: replace $w_c'' w_c'$ by w_c''/w_c'.
(3) Caption, figure 4: replace $\zeta_a s$ by ζ_a/s.
(4) Equation (A 5*b*): replace $)\overline{u}_i''$ by $\overline{u}_i'')$.